

LOGIC FOR METRIC STRUCTURES AND THE NUMBER OF UNIVERSAL SOFIC AND HYPERLINEAR GROUPS

MARTINO LUPINI

ABSTRACT. Simon Thomas gave an algebraic proof that, if the Continuum Hypothesis fails, then there are $2^{2^{\aleph_0}}$ universal sofic groups up to isomorphism, and asked if the same is true for universal hyperlinear groups. By means of model theory for metric structures, I give an alternative proof of Thomas' result, that entails the same result for universal hyperlinear groups, answering Thomas' question. As a direct consequence, I infer that the sequence of complex matrix algebras, regarded as ranked regular algebras, admits $2^{2^{\aleph_0}}$ ultraproducts under the failure of the Continuum Hypothesis, answering a question of Gábor Elek.

1. INTRODUCTION

Sofic and hyperlinear groups are two classes of discrete groups that have received the attention of many mathematicians in different areas in the last ten years. It is known that the class of sofic groups is contained in the class of hyperlinear groups, but it is not known if this containment is proper, or whether the class of hyperlinear groups is equal to the class of all discrete groups. For a complete presentation of this topic, the reader is referred to [11]. In [4], Elek and Szabó proved that a countable group Γ is sofic if and only if it can be embedded in some (or, equivalently, every) ultraproduct of the symmetric groups, regarded as a bi-invariant metric group with respect to the normalized Hamming distance (see [1] for the definition of metric ultraproducts and an introduction to logic for metric structures). An analogous characterization holds for hyperlinear groups, where the symmetric groups are replaced with the finite rank unitary groups, endowed with the normalized distance induced by the Hilbert-Schmidt norm. In view of this characterization, the metric ultraproducts of symmetric groups can be called universal sofic groups, and the metric ultraproducts of unitary groups can be called universal hyperlinear groups. In [12], Thomas proved that, if the Continuum Hypothesis fails, there are $2^{\mathfrak{c}}$ many metric ultraproducts of the symmetric groups up to (algebraic) isomorphism, where \mathfrak{c} denotes the cardinality of the continuum, and asked if the same statement holds for ultraproducts of the unitary groups. In this paper, I give a proof of Thomas' result, by means of logic for metric structures, which also applies in the case of unitary groups. From this, I deduce also the existence, under the failure of the Continuum Hypothesis, of $2^{\mathfrak{c}}$ many metric ultraproducts of the matrix algebras regarded as ranked regular rings, up to algebraic isomorphism. This problem was

2010 *Mathematics Subject Classification.* Primary 03C20 03E35 20F69; Secondary 16E50.

Key words and phrases. Ultraproducts, sofic groups, hyperlinear groups, logic for metric structures.

The research was supported by the York University Elia Scholars Program, the EST Short Visit Grant No. 4154, the National University of Singapore and the John Templeton Foundation.

raised by Gabor Elek, in view of Proposition 8.3 in [7], asserting the existence, under the failure of the Continuum hypothesis, of $2^{\mathfrak{c}}$ isomorphism classes of metric ultraproducts of the complex matrix algebras, regarded as tracial von Neumann algebras.

Under the Continuum Hypothesis, the number of metric ultraproducts of symmetric and unitary groups up to isomorphism is still unknown. In this case, the statement that they are all isomorphic is equivalent to the assertion that they are all elementarily equivalent. At the end of this article, I prove a partial result in this direction, showing that they have the same Σ_2 -theories as metric structures.

This article is organized as follows. In Section 2, I show that the sequences of symmetric and unitary groups have the order property. This implies that, under the failure of the Continuum Hypothesis, there are $2^{\mathfrak{c}}$ many ultraproducts of these sequences up to isometric isomorphism. In Section 3, I introduce some results and terminology from [7] in order to deduce the existence, under the failure of the Continuum Hypothesis, of $2^{\mathfrak{c}}$ many ultraproducts up to algebraic isomorphism. In Section 4, I infer from this result the existence, under the failure of the Continuum Hypothesis, of $2^{\mathfrak{c}}$ many ultraproducts of the sequence of matrix algebras up to algebraic isomorphism. In Section 5, I prove that all universal sofic groups are elementarily equivalent with respect to Σ_2 formulas, and that the same holds for universal hyperlinear groups.

If $n \in \mathbb{N}$, the symmetric group acting on $\{1, \dots, n\}$ is denoted by S_n and its identity by e_n . The unitary group of $n \times n$ matrices over \mathbb{C} is denoted by U_n and its identity by I_n . The symmetric group S_n is regarded as metric group with respect to the bi-invariant metric defined by

$$d_{S_n}(\sigma, \tau) = \frac{1}{n} |\{i \in \{1, \dots, n\} \mid \sigma(i) \neq \tau(i)\}|,$$

called normalized Hamming distance. The unitary group U_n is endowed with the metric

$$d_{U_n}(A, B) = \frac{\|A - B\|_2}{2\sqrt{n}}$$

where $\|\cdot\|_2$ denotes the Hilbert-Schmidt norm. Usually the factor $\frac{1}{2}$ is omitted. It is introduced here only because, in logic for bounded metric structures, for convenience all the metric spaces are supposed to have diameter at most 1. By universal sofic and, respectively, hyperlinear groups, I will mean in the following the metric ultraproducts of the sequences of the symmetric and, respectively, unitary groups. Consider, for $n \in \mathbb{N}$, the injective homomorphism $\sigma \mapsto A_\sigma$ from S_n to U_n defined by

$$(1) \quad A_\sigma(e_i) = e_{\sigma(i)}$$

for $i \in \{1, 2, \dots, n\}$, where e_1, \dots, e_n is the canonical basis of \mathbb{C}^n , and observe that

$$(2) \quad d_{U_n}(A_\sigma, A_\tau) = \sqrt{\frac{d_{S_n}(\sigma, \tau)}{2}}$$

(see [11]).

In the rest of the paper, I will use the following notational conventions. If a, b are elements of a group G , then $[a, b]$ denotes the element $aba^{-1}b^{-1}$ of G , while if a, b belong to a ring R , then the same symbol denote the element $ab - ba$ of R . Upper case calligraphic letters such as \mathcal{U}, \mathcal{V} are reserved for ultrafilters over \mathbb{N} . If $(M_n)_{n \in \mathbb{N}}$ is a sequence of metric structures and \mathcal{U} is an ultrafilter over \mathbb{N} , the

corresponding metric ultraproduct is denoted by $\prod_n^{\mathcal{U}} M_n$, while the ultrapower of a metric structure M by \mathcal{U} is denoted by $M^{\mathcal{U}}$. I will denote by \bar{x} and \bar{y} m -tuples of variables x_1, \dots, x_m and y_1, \dots, y_m . Every metric is denoted by d . The context will make clear which metric I am referring to each time. The set of natural numbers \mathbb{N} is supposed not to contain 0, and if r is a real number, then $\lceil r \rceil$ denotes the smallest integer greater than or equal to r . Finally I will write, as usual, the acronym CH to stand for “Continuum Hypothesis”.

I would like to thank my supervisor Ilijas Farah for his help and support, Samuel Coskey for his comments and suggestions, Łukasz Grabowski, Bradd Hart, Itai Ben Yacov, Ferenc Bencs, Louis-Philippe Thibault and Nigel Sequeira for many useful conversations.

2. ORDER PROPERTY FOR SYMMETRIC AND UNITARY GROUPS

In [7], Theorem 6.1, aiming to count the number of ultrapowers of a C^* -algebra or a von Neumann algebra, Farah and Shelah isolate a condition ensuring that a sequence of metric structures has 2^c many ultraproducts up to isometric isomorphism, under the failure of CH.

In this section, I will consider a particular case of [7], Theorem 6.1, for bi-invariant metric groups, and I will infer from that the following:

Proposition 1. *If CH fails and $(k_n)_{n \in \mathbb{N}}$ is a strictly increasing sequence of natural numbers, then the sequences $(S_{k_n})_{n \in \mathbb{N}}$ and $(U_{k_n})_{n \in \mathbb{N}}$ have 2^c many ultraproducts up to isometric isomorphism.*

In the following section, after introducing notations and definitions from [7], I will refine this result, showing that in this case there are in fact 2^c many ultraproducts up to algebraic isomorphism, under the failure of CH.

From [7], Theorem 5.1, considering in particular the language of bi-invariant metric groups and the formula $\varphi(x_1, x_2, y_1, y_2)$ defined by

$$d([x_1, y_2], e),$$

the following corollary can be deduced.

Corollary 2. *Let $(k_n)_{n \in \mathbb{N}}$ be a strictly increasing sequence of natural numbers and $(G_n)_{n \in \mathbb{N}}$ be a sequence of bi-invariant metric groups with uniformly bounded diameter. Suppose that, for some constant $\gamma > 0$ and every $l \in \mathbb{N}$, for all but finitely many $n \in \mathbb{N}$, G_n contains sequences $(g_{n,i})_{i=1}^l$ and $(h_{n,i})_{i=1}^l$ such that, for every $1 \leq i < j \leq l$, $g_{n,i}$ and $h_{n,j}$ commute, while if $1 \leq j \leq i \leq l$,*

$$d([g_{n,i}, h_{n,j}], e_{G_n}) \geq \gamma.$$

Then, under the failure of CH, there are 2^c many pairwise non isometrically isomorphic metric ultraproducts of the sequence $(G_{k_n})_{n \in \mathbb{N}}$.

Thus, in order to prove Proposition 1, it is enough to show that the sequences of symmetric and unitary groups satisfy the hypothesis of Corollary 2.

Lemma 3. *For every $n \in \mathbb{N}$, there are sequences $(\sigma_{n,i})_{i=1}^n, (\tau_{n,i})_{i=1}^n$ of elements of S_{3^n} such that, if $1 \leq i < j \leq n$, $\sigma_{n,i}$ and $\tau_{n,j}$ commute, while, if $1 \leq j \leq i \leq n$, $[\sigma_{n,i}, \tau_{n,j}]$ is the product of 3^{n-1} disjoint cycles of length 3*

Proof. If $n \in \mathbb{N}$, consider the action of $\overbrace{S_3 \times \dots \times S_3}^{n \text{ times}}$ on $\{1, 2, 3\}^n$ defined by

$$(\sigma_1, \dots, \sigma_n)(i_1, \dots, i_n) = (\sigma_1(i_1), \dots, \sigma_n(i_n)),$$

where $\sigma_1, \dots, \sigma_n \in S_3$ and $i_1, \dots, i_n \in \{1, 2, 3\}$. This action defines an isometric embedding

$$(\sigma_1, \dots, \sigma_n) \rightarrow \alpha(\sigma_1, \dots, \sigma_n)$$

of $\overbrace{S_3 \times \dots \times S_3}^{n \text{ times}}$ into the group of permutations of $\{1, 2, 3\}^n$, which we identify with S_{3^n} . Define also, for $i = 1, \dots, n$,

$$\sigma_{n,i} = \alpha \left(\overbrace{(12), \dots, (12)}^{i \text{ times}}, \overbrace{e_3, \dots, e_3}^{n-i \text{ times}} \right)$$

and

$$\tau_{n,i} = \alpha \left(\overbrace{e_3, \dots, e_3}^{i-1 \text{ times}}, (23), \overbrace{e_3, \dots, e_3}^{n-i \text{ times}} \right).$$

Observe that, for $i < j$,

$$[\sigma_{n,i}, \tau_{n,j}] = e_{3^m},$$

while, for $i \geq j$,

$$[\sigma_{n,i}, \tau_{n,j}] = \alpha \left(\overbrace{e_3, \dots, e_3}^{j-1 \text{ times}}, (123), \overbrace{e_3, \dots, e_3}^{n-j \text{ times}} \right). \quad \square$$

Lemma 4. *If $n, k, l \in \mathbb{N}$ and $r \in \mathbb{N} \cup \{0\}$ are such that $n = 3^l k + r$ and $0 \leq r < 3^l$, there are sequences $(\Sigma_{n,i})_{i=1}^l$ and $(T_{n,i})_{i=1}^l$ in S_n such that, for $1 \leq i, j \leq l$, $\Sigma_{n,i}$ and $T_{n,i}$ commute if $i < j$, while $[\Sigma_{n,i}, T_{n,j}]$ is the product of $3^{l-1}k$ disjoint cycles of length 3 and, in particular,*

$$d([\Sigma_{n,i}, T_{n,j}], e) = \frac{3^l k}{3^l k + r} \geq \frac{k}{k+1} \geq \frac{1}{2},$$

if $i \geq j$.

Proof. Consider the action of S_{3^l} on $\{1, \dots, 3^l\} \times \{1, \dots, k\}$ defined by

$$\sigma(i, j) = (\sigma(i), j)$$

for every $i \in \{1, \dots, 3^l\}$ and $j \in \{1, \dots, k\}$. This defines an isometric embedding of S_{3^l} into $S_{k \cdot 3^l}$. Moreover, letting $S_{3^l k}$ act on the first $3^l k$ elements of $\{1, \dots, n\}$ defines an algebraic embedding of $S_{3^l k}$ into S_n . The composition Φ of these two embeddings is an algebraic embedding of S_{3^l} into S_n . Define

$$\Sigma_{n,i} = \Phi(\sigma_{l,i})$$

and

$$T_{n,i} = \Phi(\tau_{l,i})$$

for $1 \leq i \leq l$. Then, if $1 \leq i, j \leq l$,

$$[\Sigma_{n,i}, T_{n,j}] = [\Phi(\sigma_{l,i}), \Phi(\tau_{l,j})] = \Phi([\sigma_{l,i}, \tau_{l,j}]).$$

If $i < j$, $[\sigma_{l,i}, \tau_{l,j}]$ is the identity and, hence, $[\Sigma_{n,i}, T_{n,j}]$ is the identity. If $i \geq j$ then $[\sigma_{l,i}, \tau_{l,j}]$ is a product of 3^{l-1} disjoint 3-cycles and, hence, $[\Sigma_{n,i}, T_{n,j}]$ is the product of $3^{l-1}k$ disjoint 3-cycles. \square

Corollary 5. *If $n, k, l \in \mathbb{N}$ and $r \in \mathbb{N} \cup \{0\}$ are as in the statement of Lemma 4, then there are sequences $(b_{n,i})_{i=1}^l$ and $(c_{n,i})_{i=1}^l$ in U_n such that $b_{n,i}$ and $c_{n,i}$ commute if $i < j$, while $d([b_{n,i}, c_{n,j}]) \geq \frac{1}{2}$ if $i \geq j$.*

Proof. I simply define, as in 1,

$$b_{n,i} = A_{\Sigma_{n,i}}$$

and

$$c_{n,i} = A_{T_{n,i}}.$$

Observe that if $i < j$, then

$$[\Sigma_{n,i}, \Sigma_{n,j}] = e_n$$

and

$$[A_{\Sigma_{n,i}}, A_{T_{n,j}}] = A_{[\Sigma_{n,i}, T_{n,j}]} = A_{e_n} = I_n,$$

while, if $i \geq j$,

$$d([\Sigma_{n,i}, T_{n,i}], e) \geq \frac{1}{2}$$

and, by 2,

$$d([A_{\Sigma_{n,i}}, A_{T_{n,j}}], I_n) = \sqrt{\frac{d([\Sigma_{n,i}, T_{n,j}], e_n)}{2}} \geq \frac{1}{2}. \quad \square$$

Proposition 1 follows now directly from Lemma 4 and its corollary, applying Corollary 2.

3. NON-ISOMORPHIC UNIVERSAL SOFIC AND HYPERLINEAR GROUPS

In this section, I refine Proposition 1 and prove the following

Theorem 6. *If CH fails and $(k_n)_{n \in \mathbb{N}}$ is an increasing sequence of natural numbers, then, up to algebraic isomorphism, there are 2^c many ultraproducts of both the sequence $(S_{k_n})_{n \in \mathbb{N}}$ and the sequence $(U_{k_n})_{n \in \mathbb{N}}$.*

This result has already been proved by Thomas in [12] for permutation groups. Lukasz Grabowski pointed out to me that the case of unitary groups can be deduced from Proposition 8.3 in [7], using the facts that non-isomorphic II_1 factors have non-isomorphic unitary groups (see [2]) and that the unitary group of an ultraproduct of factors is the ultraproduct of the unitary groups ([8], Proposition 2.1). In the following, I will give a direct proof of this result by means of logic for metric structures. This generalization of usual (discrete) logic is suitable to deal with structures endowed with a metric. For an introduction to this topic, the reader is referred to [1]. Structures and formulas in usual discrete logic can be considered particular cases of metric structures and formulas, where the distance is interpreted as the trivial discrete distance defined by $d(x, y) = 1$ iff $x \neq y$. Thus, definitions and theorems stated in the setting of logic for metric structures subsume the analogous definitions and theorems for usual discrete logic as a particular case. For the sake of simplicity, all the languages are henceforth supposed without relation symbols, apart from the metric. If M is a structure in such a language \mathcal{L} , denote by M_{alg} the \mathcal{L} -structure obtained by M replacing the metric on M by the trivial discrete metric.

I have now to introduce some notation and recall some results from [7]. If \mathcal{L} is a language, $\psi(\bar{x}, \bar{y})$ is an \mathcal{L} -formula, $\varepsilon \geq 0$ and M is an \mathcal{L} -structure, the relation $\prec_{\psi, \varepsilon}$ on M^k is defined by

$$\bar{a} \prec_{\psi, \varepsilon} \bar{b} \Leftrightarrow (\psi^M(\bar{a}, \bar{b}) \leq \varepsilon \wedge \psi^M(\bar{b}, \bar{a}) \geq 1 - \varepsilon).$$

A chain in M^k with respect to the relation $\prec_{\psi, \varepsilon}$ will be called a (ψ, ε) -chain in M . The relation $\prec_{\psi, 0}$ will be denoted by \prec_ψ and a $(\psi, 0)$ -chain will be called a ψ -chain. If M is an \mathcal{L} -structure and $\varphi(\bar{x}, \bar{y})$ an \mathcal{L} -formula, a ψ -chain \mathcal{C} is called (\aleph_1, ψ) -skeleton like if, for every $\bar{a} \in M^k$ there is a countable $\mathcal{C}_{\bar{a}} \subset \mathcal{C}$ such that, for every $\bar{b}, \bar{c} \in \mathcal{C}$ such that

$$\{x \in \mathcal{C}_{\bar{a}} \mid \bar{b} \preceq_\psi x \preceq_\psi \bar{c}\} = \emptyset,$$

one has

$$\psi^M(\bar{a}, \bar{b}) = \psi^M(\bar{a}, \bar{c}) \quad \text{and} \quad \psi^M(\bar{b}, \bar{a}) = \psi^M(\bar{c}, \bar{a}).$$

The notion of ψ -chain and (\aleph_1, ψ) -skeleton like ψ -chain in a discrete structure for a discrete formula ψ are obtained from the previous ones, as a particular case.

The following statement is proved in [7] (Proposition 4.2 and 6.6).

Lemma 7. *If $\varphi(\bar{x}, \bar{y})$ is an \mathcal{L} -formula, I is a linear order of cardinality \mathfrak{c} and $(M_n)_{n \in \mathbb{N}}$ is a sequence of \mathcal{L} -structures such that, $\forall n \in \mathbb{N}$, M_n contains a φ -chain of length n , then there is an ultrafilter \mathcal{U} over \mathbb{N} such that $\prod_n^{\mathcal{U}} M_n$ contains an (\aleph_1, φ) -skeleton like φ -chain of order type I .*

The same fact for discrete structures and formulas follows from this lemma as a particular case. It is easy to see that the proof of this proposition can be modified to obtain the same conclusion under the following slightly weaker assumption: $\forall \varepsilon > 0$, $\forall l \in \mathbb{N}$, $\exists n_0 \in \mathbb{N}$ such that, $\forall n \geq n_0$, M_n contains a (φ, ε) -chain of length l . If a sequence $(M_n)_{n \in \mathbb{N}}$ of \mathcal{L} -structures satisfies this assumption, I will say, following [5], that the sequence $(M_n)_{n \in \mathbb{N}}$ has the *order property* witnessed by the \mathcal{L} -formula $\varphi(\bar{x}, \bar{y})$.

The connection between the number of non-isomorphic ultraproducts and the (\aleph_1, φ) -skeleton like φ -chains is given by the following lemma, proved in [7] (Proposition 3.14).

Lemma 8. *If CH fails, $\varphi(\bar{x}, \bar{y})$ is an \mathcal{L} -formula and \mathcal{K} is a class of \mathcal{L} -structures such that, for every linear order I of cardinality \mathfrak{c} , there is an element M of \mathcal{K} such that M contains an (\aleph_1, φ) -skeleton like φ -chain, then there are $2^\mathfrak{c}$ many pairwise non-isometrically isomorphic \mathcal{L} -structures in \mathcal{K} .*

The following lemma is useful when, as in our case, one is interested in counting the number of metric ultraproducts up to algebraic isomorphism.

Lemma 9. *Suppose that $\varphi(\bar{x}, \bar{y})$ is an \mathcal{L} -formula and $\psi(\bar{x}, \bar{y})$ is a discrete \mathcal{L} -formula such that, for every \mathcal{L} -structure M , for every $\bar{a}, \bar{b} \in M^k$, $\psi(\bar{a}, \bar{b})$ holds in M_{alg} if and only if $\varphi^M(\bar{a}, \bar{b}) = 0$. If M is an \mathcal{L} -structure and \mathcal{C} is an (\aleph_1, φ) -skeleton like φ -chain in M , then \mathcal{C} is an (\aleph_1, ψ) -skeleton like ψ -chain of the same order type in M_{alg} .*

Proof. The hypothesis implies that \prec_ψ in M_{alg}^k refines \prec_φ in M^k . Thus, a φ -chain in M is also a ψ -chain in M_{alg} of the same order type. Moreover, suppose $\bar{a} \in M^k$ and $\mathcal{C}_{\bar{a}}$ is as in the definition of (\aleph_1, φ) -skeleton like. If $\bar{b}, \bar{c} \in \mathcal{C}$ are such that

$$\{x \in \mathcal{C}_{\bar{a}} \mid \bar{b} \preceq_\psi x \preceq_\psi \bar{c}\} = \emptyset,$$

then also

$$\{x \in \mathcal{C}_{\bar{a}} \mid \bar{b} \preceq_{\varphi} x \preceq_{\varphi} \bar{c}\} = \emptyset.$$

Hence,

$$\varphi^M(\bar{a}, \bar{b}) = \varphi^M(\bar{a}, \bar{c}) \quad \text{and} \quad \varphi^M(\bar{b}, \bar{a}) = \varphi^M(\bar{c}, \bar{a}).$$

Since, by hypothesis, $M_{alg} \models \psi^M(\bar{a}, \bar{b})$ is equivalent to $\varphi^M(\bar{a}, \bar{b}) = 0$ and $M_{alg} \models \psi^M(\bar{a}, \bar{c})$ is equivalent to $\varphi^M(\bar{a}, \bar{c}) = 0$, one gets

$$M_{alg} \models (\psi^M(\bar{a}, \bar{b}) \leftrightarrow \psi^M(\bar{a}, \bar{c})).$$

In the same way,

$$M_{alg} \models (\psi^M(\bar{b}, \bar{a}) \leftrightarrow \psi^M(\bar{c}, \bar{a}))$$

is deduced. Thus, \mathcal{C} is an (\aleph_1, ψ) -skeleton like ψ -chain in M_{alg} . \square

Remark 10. If s, t are \mathcal{L} -terms and $q : [0, 1] \rightarrow [0, 1]$ is a continuous function such that $q(x) = 0$ iff $x = 0$, the formulas

$$\varphi(s, t) = "q(d(s, t))" \quad \text{and} \quad \psi(s, t) = "s = t"$$

satisfy the hypothesis of the previous lemma.

From Lemmas 7, 8 and 9, the following proposition can be deduced:

Proposition 11. Assume that CH fails. If $\varphi(\bar{x}, \bar{y})$ and $\psi(\bar{x}, \bar{y})$ are as in Lemma 9, $(M_n)_{n \in \mathbb{N}}$ is a sequence of \mathcal{L} -structures with the order property witnessed by φ and $(k_n)_{n \in \mathbb{N}}$ is a strictly increasing sequence of natural numbers, then the family of \mathcal{L} -structures

$$\left\{ \left(\prod_n^{\mathcal{U}} M_{k_n} \right)_{alg} \mid \mathcal{U} \text{ is an ultrafilter over } \mathbb{N} \right\},$$

where $\prod_n^{\mathcal{U}} M_{k_n}$ denotes the metric ultraproduct, contains $2^{\mathfrak{c}}$ many pairwise non-isomorphic elements. In other words, there are $2^{\mathfrak{c}}$ many metric ultraproducts of the sequence $(M_{k_n})_{n \in \mathbb{N}}$ up to algebraic isomorphism, i.e., up to bijections preserving all the function symbols but not necessarily the distance.

Proof. Since every subsequence of $(M_n)_{n \in \mathbb{N}}$ has the order property witnessed by φ , there is no loss of generality assuming $k_n = n$ for every $n \in \mathbb{N}$. By Lemma 7, for every linear order I of cardinality \mathfrak{c} , there is an ultrafilter \mathcal{U} such that $\prod_n^{\mathcal{U}} M_n$ has an (\aleph_1, φ) -skeleton like φ -chain \mathcal{C} of order type I . By Lemma 9, \mathcal{C} is also an (\aleph_1, ψ) -skeleton like ψ -chain of the same order type in $\left(\prod_n^{\mathcal{U}} M_n \right)_{alg}$. Since this is true for every linear order I of cardinality \mathfrak{c} , the conclusion follows from Lemma 8. \square

Observe that in this result $2^{\mathfrak{c}}$ is the maximum number possible, because it is the number of ultrafilters over \mathbb{N} . Remark 10 allows me to formulate the following corollary, which is a particular case of Proposition 11.

Corollary 12. If $(M_n)_{n \in \mathbb{N}}$ is a sequence of \mathcal{L} -structures with the order property witnessed by the \mathcal{L} -formula $q(d(s, t))$, where s and t are terms and $q : [0, 1] \rightarrow [0, 1]$ is a continuous function such that $q(x) = 0$ iff $x = 0$, then the conclusion of Proposition 11 holds.

Now, in order to prove Theorem 6 it is enough to show that the sequences $(S_n)_{n \in \mathbb{N}}$ and $(U_n)_{n \in \mathbb{N}}$ have the order property, witnessed by a formula φ as in Corollary 12. Let $\eta(x_1, x_2, y_1, y_2)$ be the metric formula defined by

$$\min \{2d([x_1, y_2], e), 1\}.$$

Fix $l \in \mathbb{N}$ and $n \geq 3^l$, and consider the sequences $(\Sigma_{n,i})_{i=1}^l$ and $(T_{n,i})_{i=1}^l$ in S_n defined in Lemma 4 and the sequences $(b_{n,i})_{i=1}^l$ and $(c_{n,i})_{i=1}^l$ in U_n defined in Corollary 5. It is not difficult to infer from Lemma 4 and Corollary 5 that

$$((\Sigma_{n,i}, T_{n,i}))_{i=1}^l \quad \text{and} \quad ((b_{n,i}, c_{n,i}))_{i=1}^l$$

are η -chains of length l in S_n and U_n respectively. An application of Corollary 12 concludes the proof of Theorem 6.

4. RANKED REGULAR RINGS

If $n \in \mathbb{N}$, denote by \mathbb{M}_n the algebra of $n \times n$ matrices over \mathbb{C} and by rk the normalized rank on \mathbb{M}_n . Thus, if $A \in \mathbb{M}_n$, $\text{rk}(A)$ is the rank of A divided by n . In [3], Elek consider metric ultraproducts of the matrix algebras over \mathbb{C} with respect to the metric

$$d(a, b) = \text{rk}(a - b)$$

induced by the rank. In [7] it is shown that if CH fails, the sequence of matrix algebras regarded as tracial von Neumann algebras has $2^{\mathfrak{c}}$ many pairwise non-isomorphic metric ultraproducts. In this section, I prove that the same is true considering metric ultraproducts as in [3].

If $\sigma \in S_n$, denote, as before, by A_σ the permutation matrix associated to σ , regarded as an element of \mathbb{M}_n .

Lemma 13. *If $\sigma \in S_n$ is the product of l possibly trivial disjoint cycles, then*

$$\text{rk}(I_n - A_\sigma) = 1 - \frac{l}{n}.$$

Proof. The proof is by induction on l . If $l = 1$ then, since the statement is conjugation invariant, I can assume $\sigma = (12 \dots n)$. It is clear that, in this case, $\text{rk}(I - A_\sigma) = \frac{n-1}{n}$. If $l > 1$ then, again by conjugation invariance, I can assume that there is $m < n$ such that $\{0, 1, \dots, m-1\}$ and $\{m, \dots, n-1\}$ are σ -invariant sets. Thus, there are $\tau_1 \in S_m$ and $\tau_2 \in S_{n-m}$ product of l_1 and l_2 disjoint possibly trivial cycles respectively such that $A_\sigma = A_{\tau_1} \oplus A_{\tau_2}$ and $l = l_1 + l_2$. Therefore, by the induction hypothesis,

$$\begin{aligned} \text{rk}(I_n - A_\sigma) &= \text{rk}((I_m - A_{\tau_1}) \oplus (I_{n-m} - A_{\tau_2})) \\ &= \frac{m \cdot \text{rk}(I_m - A_{\tau_1}) + (n-m) \cdot \text{rk}(I_{n-m} - A_{\tau_2})}{n} \\ &= \frac{n - l_1 - l_2}{n} \\ &= \frac{n - l}{n}. \end{aligned}$$

This concludes the proof by induction. \square

Lemma 14. *If $n, k, l, r \in \mathbb{N}$ are such that $n = 3^l k + r$ and $0 \leq r < 3^l$, then there are sequences $(B_{n,i})_{i=1}^l$ and $(C_{n,i})_{i=1}^l$ in \mathbb{M}_n such that $[B_{n,i}, C_{n,j}] = 0$ for $i < j$ and $\text{rk}([B_{n,i}, C_{n,j}]) \geq \frac{1}{3}$ for $i \geq j$.*

Proof. Consider the sequence $(\Sigma_{n,i})_{i=1}^l$ and $(T_{n,i})_{i=1}^l$ in S_n as defined in the previous section, and define $B_{n,i} = A_{\Sigma_{n,i}}$ and $C_{n,i} = A_{T_{n,i}}$. Observe that

$$\text{rk}([B_{n,i}, C_{n,j}]) = \text{rk}(I_n - A_{[\Sigma_{n,i}, T_{n,j}]}) .$$

In particular, if $i < j$ then $[\Sigma_{n,i}, T_{n,i}] = e$ and $[B_{n,i}, C_{n,j}] = 0$, while if $i \geq j$ then $[\Sigma_{n,i}, T_{n,j}]$ is a product of $3^{l-1}k$ disjoint 3-cycles. From the previous lemma, it follows that

$$\text{rk}([B_{n,i}, C_{n,j}]) = \frac{k3^l - k3^{l-1}}{k3^l + r} \geq \frac{2k3^{l-1}}{(k+1)3^l} \geq \frac{1}{3}. \quad \square$$

Proposition 15. *If CH fails, then for every increasing sequence $(k_n)_{n \in \mathbb{N}}$ of natural numbers there are 2^c many metric ultraproducts of the sequence $(\mathbb{M}_{k_n})_{n \in \mathbb{N}}$ with respect to the metric induced by the rank whose multiplicative semigroups are pairwise non-isomorphic*

Proof. For every $n \in \mathbb{N}$, regard \mathbb{M}_n as a metric structure in the language \mathcal{L} containing only one binary function, interpreted as the product, and the metric symbol, interpreted as the metric induced by the rank. Consider the \mathcal{L} -formula $\varphi(x_1, x_2, y_1, y_2)$ defined by

$$\min\{3d(x_1y_2, y_2x_1), 1\}$$

and observe that, by the previous lemma, \mathbb{M}_n has a φ -chain of length l for every $n \geq 3^l$. The conclusion now follows from Corollary 12. \square

The previous theorem can be in fact generalized to direct sequences and direct limits of finite products of matrix algebras obtained from a Bratteli diagram and a harmonic function as in [3], Section 3. In the following, I will use the terminology and notation of this reference.

Theorem 16. *Suppose that $(A_n)_{n \in \mathbb{N}}$ is a direct sequence of algebras with direct limit A , obtained from a Bratteli diagram $\mathcal{B} = (Z_n)_{n \in \mathbb{N}}$ with weight function W and a harmonic function $(P_n)_{n \in \mathbb{N}}$, such that, for some $\gamma > 0$, for every $m \in \mathbb{N}$ there is $n \in \mathbb{N}$ such that*

$$\sum \{P_n(\alpha) \mid \alpha \in Z_n, W(\alpha) \geq m\} \geq \gamma,$$

then the sequence $(A_n)_{n \in \mathbb{N}}$ (resp. A) has 2^c many ultraproducts (resp. ultrapowers over \mathbb{N}) whose multiplicative semigroups are pairwise non-isomorphic.

Proof. Fix $l \in \mathbb{N}$. For every $n \in \mathbb{N}$, define

$$r_n = \sum \{P_n(\alpha) \mid \alpha \in Z_n, W(\alpha) \geq l\}.$$

Observe that $(r_n)_{n \in \mathbb{N}}$ is a monotone increasing sequence in $[0, 1]$ such that, for some $n_0 \in \mathbb{N}$, $r_{n_0} \geq \gamma$. For $n \geq n_0$ and $\alpha \in Z_n$ such that $W(\alpha) \geq l$, define the sequences $(B_i^\alpha)_{i=1}^l$ and $(C_i^\alpha)_{i=1}^l$ in $\mathbb{M}_{W(\alpha)}$ as in Lemma 14, whilst set $B_i^\alpha = C_i^\alpha = I_{W(\alpha)}$ if $W(\alpha) < l$. If $i \in \{1, 2, \dots, l\}$, define

$$\mathbf{B}_i = (B_i^\alpha)_{\alpha \in Z_n} \quad \text{and} \quad \mathbf{C}_i = (C_i^\alpha)_{\alpha \in Z_n}.$$

Observe that, if $i < j$,

$$[\mathbf{B}_i, \mathbf{C}_j] = 0$$

while if $i \geq j$,

$$\text{rk}([\mathbf{B}_i, \mathbf{C}_j]) \geq \frac{\gamma}{3}.$$

Thus, if $\varphi(x_1, x_2, y_1, y_2)$ is the formula

$$\min \left\{ \frac{3}{\gamma} d(x_1 y_2, y_2 x_1), 1 \right\},$$

the sequence $((\mathbf{B}_i, \mathbf{C}_i))_{i=1}^l$ is a φ -chain in A_n and (since φ is quantifier-free) in A as well. The conclusion is now inferred from Corollary 12 as in the proof of the previous proposition. \square

Observe that all the statements hold without change and with same proof if \mathbb{C} is replaced by any other field or by any von Neumann regular ring R endowed with a rank function N . In this latter case, the rank function considered on the ring $M_n(R)$ of $n \times n$ matrices over R is the only rank function N_n such that, if $\Delta(a)$ is the diagonal matrix with all nonzero entries equal to a , $N_n(\Delta(a)) = N(a)$ (see Corollary 16.10 in [10]). An exhaustive treatment of von Neumann regular rings and ranked von Neumann regular rings can be found in [10].

From this observation, one can deduce the following observation, that can be regarded as an algebraic analogue of the fact any non type I von Neumann algebra has 2^c many pairwise non-isomorphic tracial ultrapowers, if CH fails. An account of type classification for regular right self-injective can be found, again, in [10].

Proposition 17. *If CH fails, then any non type I_f right self-injective von Neumann regular ring R has 2^c many ultrapowers whose multiplicative semigroups are pairwise non-isomorphic*

Proof. By Theorem 10.24, Theorem 7.20 and Lemma 7.17 from [10], if R is a non type I von Neumann regular ring, then for every $n \in \mathbb{N}$, R has a subring J isomorphic to $M_n(S)$ for some regular ring S . Define the (discrete) formula $\varphi(x_1, x_2, y_1, y_2)$ by

$$x_1 y_2 = y_2 x_1.$$

If $n \geq 3^l$, the sequences of elements of J defined in Lemma 14 give a φ -chain of length l in R . Since φ contains only the multiplication function symbol, the conclusion can be deduced from [7], Theorem 5.1, as in the proof of Proposition 15. \square

5. THE Σ_2 -THEORIES OF UNIVERSAL SOFIC AND HYPERLINEAR GROUPS

If CH holds, all the universal sofic and hyperlinear groups are saturated (see [1]), and hence (isometrically) isomorphic if and only if their (metric) theories coincide. It is not known if there are two algebraically non-isomorphic sofic or hyperlinear groups. It is asked in [12] if all the universal sofic groups are elementarily equivalent. I prove here a partial result, namely that universal sofic groups, as well as universal hyperlinear groups, have all the same metric Σ_2 -theories. This is equivalent to the statement that for any Σ_2 formula φ in the language of metric groups, the sequences of real numbers given by evaluation of φ in the symmetric groups and, respectively, in the unitary groups, converge. Since a formula φ is Π_2 iff $1 - \varphi$ is Σ_2 , this implies that universal sofic, and respectively hyperlinear, groups have the same Π_2 theories as well.

A function $\iota : M \rightarrow N$ between structures in a metric language is said to preserve all the function and relation symbols up to $\delta \geq 0$ if, for every n -ary function symbol f and $a_1, \dots, a_n \in M$,

$$d^N(f^N(\iota(a_1), \dots, \iota(a_n)), \iota(f^M(a_1, \dots, a_n))) \leq \delta$$

and for every n -ary relation symbol R and $a_1, \dots, a_n \in M$,

$$|R^N(\iota(a_1), \dots, \iota(a_n)) - R^M(a_1, \dots, a_n)| \leq \delta.$$

Lemma 18. *Assume that $(M_n)_{n \in \mathbb{N}}$ is a sequence of structures in a metric language \mathcal{L} . Suppose that, $\forall \delta > 0$, $\exists m_0 \in \mathbb{N}$ such that, $\forall m \geq m_0$ $\exists k_0 \in \mathbb{N}$ such that $\forall k \geq k_0$ there is an embedding ι_m^k of M_m into M_k satisfying the following properties:*

- ι_m^k preserves all the relation and function symbols up to δ
- $\forall x \in M_k \exists y \in M_m$ with $d(\iota_m^k(y), x) < \delta$.

It follows that, if ψ is a quantifier-free \mathcal{L} -formula and φ is the \mathcal{L} -formula

$$\inf_x \sup_{y_1, \dots, y_n} \psi(y_1, \dots, y_n, x),$$

then the sequence $(\varphi^{M_m})_{m \in \mathbb{N}}$ converge.

Proof. Suppose $\varepsilon > 0$ and $\delta > 0$ are such that any embedding preserving all the function and relation symbols up to δ preserves ψ up to ε . If ι_m^k is as in the statement, then for every $a \in M_k$ there is $\tilde{a} \in M_m$ such that $d(a, \iota_m^k(\tilde{a})) < \delta$. Hence,

$$\begin{aligned} \sup_{y_1, \dots, y_n \in M_k} \psi(y_1, \dots, y_n, a) &\geq \sup_{y_1, \dots, y_n \in M_k} \psi(y_1, \dots, y_n, \iota_m^k(\tilde{a})) - \varepsilon \\ &\geq \sup_{y_1, \dots, y_n \in M_m} \psi(\iota_m^k(y_1), \dots, \iota_m^k(y_n), \iota_m^k(\tilde{a})) - \varepsilon \\ &\geq \sup_{y_1, \dots, y_n \in M_m} \psi(y_1, \dots, y_n, \tilde{a}) - 2\varepsilon \\ &\geq \inf_{x \in M_m} \sup_{y_1, \dots, y_n \in M_m} \psi(y_1, \dots, y_n, x) - 2\varepsilon \\ &= \varphi^{M_m} - 2\varepsilon \end{aligned}$$

Since this is true for every $a \in M_k$,

$$\varphi^{M_k} = \inf_{x \in M_k} \sup_{y_1, \dots, y_n \in M_k} \psi(y_1, \dots, y_n, x) \geq \varphi^{M_m} - 2\varepsilon.$$

Since this is true for every $k \geq k_0$,

$$\liminf_{k \rightarrow +\infty} \varphi^{M_k} \geq \varphi^{M_m} - 2\varepsilon$$

and since this is true for every $m \geq m_0$,

$$\liminf_{k \rightarrow +\infty} \varphi^{M_k} \geq \limsup_{m \rightarrow +\infty} \varphi^{M_m} - 2\varepsilon.$$

Finally, letting ε go to 0, one gets

$$\liminf_{k \rightarrow +\infty} \varphi^{M_k} \geq \limsup_{m \rightarrow +\infty} \varphi^{M_m}.$$

This concludes the proof. \square

The following result is proved in [13], using the spectral theorem for normal matrices and an averaging argument on the eigenvalues.

Proposition 19. *If $\varepsilon > 0$, there is $m_0 \in \mathbb{N}$ such that, for every $k \in \mathbb{N}$ and $m \geq m_0$, if $A \in \mathbb{M}_{km}$ is a normal matrix with operator norm at most 1, there is $B \in M_m$ of operator norm at most 1 and $W \in U_{km}$ such that $\|A - W(I_k \otimes B)W^*\|_2 < \varepsilon$, where \otimes denotes the usual tensor product of matrices and $\|\cdot\|_2$ the normalized Hilbert-Schmidt norm. Moreover, if A is Hermitian (resp. unitary), B can be chosen Hermitian (resp. unitary).*

Observe that if $W \in U_{km}$, the function from U_m to U_{km} sending B to

$$W(I_k \otimes B)W^*$$

is an isometric embedding.

Lemma 20. *If $k, m \in \mathbb{N}$ and $r \in \{0, 1, \dots, m-1\}$, then the function ι from U_{km} to U_{km+r} sending A to*

$$\begin{pmatrix} A & 0 \\ 0 & I_r \end{pmatrix}$$

(where I_r is the $r \times r$ identity matrix) is an algebraic embedding that preserves the metric up to $\frac{1}{k}$ and whose image is $\frac{4}{\sqrt[4]{k}}$ -dense in U_{km+r} .

Proof. Denote $km + r$ by n . Direct calculation shows that, if $A, B \in U_{km}$,

$$0 \leq d(A, B) - d(\iota(A), \iota(B)) = 1 - \sqrt{\frac{km}{n}} \leq \frac{1}{k}.$$

Suppose now that $C \in U_n$ and define A to be the element of \mathbb{M}_{km} such that $A_{i,j} = C_{i,j}$ for $1 \leq i, j \leq km$ and $A_{i,j} = 0$ for $\max\{i, j\} > km$. It is easy to see that, since U is unitary,

$$\|AA^* - I_{km}\|_2^2 \leq \frac{1}{k}.$$

By [9] (Corollary 1), there is $B \in U_{km}$ such that

$$\|A - B\|_2^2 \leq 36\sqrt{\|AA^* - I\|} \leq \frac{36}{\sqrt{k}}.$$

Thus,

$$d(\iota(B), C) \leq \frac{1}{2} \sqrt{\frac{km\|A - B\|_2^2 + 2r}{n}} \leq \frac{4}{\sqrt[4]{k}}. \quad \square$$

Theorem 21. *If φ is a Σ_2 formula of the form*

$$\inf_x \sup_{y_1, \dots, y_n} \varphi(y_1, \dots, y_n, x)$$

in the language of bi-invariant metric groups, then the sequence $(\varphi^{U_n})_{n \in \mathbb{N}}$ converges.

Proof. The sequence of unitary groups satisfy the hypothesis of Lemma 18 by Lemma 20 and Proposition 19. \square

In order to show that the same holds for symmetric groups, it is enough to prove the analogue of von Neumann's result in this setting.

If $\sigma \in S_n$ and $k \geq 1$, define $C_k(\sigma)$ to be the set of cycles of σ of length k and $w(\sigma)$ the smallest k such that $C_{k+1}(\sigma) = \emptyset$. In particular, $C_1(\sigma)$ is the set of fixed points of σ .

Lemma 22. *For every $m, n \in \mathbb{N}$ such that $m|n$, if $\sigma \in S_n$, then there is $\tau \in S_n$ such that $w(\tau) \leq m$, $C_1(\tau) \supset C_1(\sigma)$ and $d(\sigma, \tau) \leq \frac{2}{m}$.*

Proof. Suppose $k \in \mathbb{N}$ is such that $n = km$. If σ is a cycle of length n , there is a product of k cycles of length m at distance $\frac{1}{m}$ from σ . Suppose σ has no cycle of length n . If $m < l < n$, then $l = \lambda m + \rho$ for some $0 \leq \rho < m$ and $1 \leq \lambda < k$.

Pick $c \in C_l(\sigma)$ and consider the permutation σ' obtained by σ breaking up c into λ cycles of length m and, if $\rho > 0$, one cycle of length ρ . Thus,

$$km \cdot d(\sigma, \sigma') \leq \lambda + 1.$$

Consider the permutation τ obtained by σ repeating this process for any element of $\bigcup_{k \geq m} C_k(\sigma)$. Then, $w(\tau) \leq m$. Define, for $\lambda \in \{1, 2, \dots, k-1\}$ and $\rho \in \{0, 1, \dots, m-1\}$,

$$n_{\lambda, \rho} = |C_{\lambda m + \rho}(\sigma)|.$$

Observe that

$$m \sum_{\lambda=1}^{k-1} \lambda \sum_{\rho=0}^{m-1} n_{\lambda, \rho} \leq \sum_{\lambda=1}^{k-1} \sum_{\rho=0}^{m-1} (\lambda m + \rho) n_{\lambda, \rho} \leq n = km,$$

and hence

$$\sum_{\lambda=1}^{k-1} \sum_{\rho=0}^{m-1} \lambda n_{\lambda, \rho} \leq k.$$

Now, we have

$$km \cdot d(\sigma, \tau) \leq \sum_{\lambda=1}^{k-1} \sum_{\rho=0}^{m-1} (\lambda + 1) n_{\lambda, \rho} \leq 2 \sum_{\lambda=1}^{k-1} \sum_{\rho=0}^{m-1} \lambda n_{\lambda, \rho} \leq 2k$$

and hence $d(\sigma, \tau) \leq \frac{2}{m}$. \square

Corollary 23. *For every $m, n \in \mathbb{N}$ such that $m|n$ and $\beta \in (0, 1)$, if $\sigma \in S_n$, then there is $\tau \in S_n$ such that $C_1(\tau) \supset C_1(\sigma)$, $w(\tau) \leq \lceil m^\beta \rceil$ and $d(\sigma, \tau) \leq \frac{8}{m^\beta}$.*

Proof. Define $N = \lceil \frac{km}{m^\beta} \rceil \lceil m^\beta \rceil$ and observe that

$$N \leq \left(\frac{km}{m^\beta} + 1 \right) (m^\beta + 1) = km + \frac{km}{m^\beta} + m^\beta + 1 \leq 4km.$$

If $\sigma \in S_{km}$, consider the element $\tilde{\sigma}$ of S_N acting as σ on $\{1, 2, \dots, km\}$ and fixing $\{km+1, \dots, N\}$ pointwise. By the previous proposition, there is $\tilde{\tau} \in S_N$ such that $C_1(\tilde{\tau}) \supset C_1(\tilde{\sigma})$, $w(\tilde{\tau}) \leq \lceil m^\beta \rceil$ and $d(\tilde{\sigma}, \tilde{\tau}) \leq \frac{2}{\lceil m^\beta \rceil}$. Consider now the element τ of S_{km} obtained restricting $\tilde{\tau}$ to $\{1, 2, \dots, km\}$. Now, $w(\tau) \leq \lceil m^\beta \rceil$, $C_1(\tau) \supset C_1(\sigma)$ and

$$d(\sigma, \tau) = \frac{N}{km} d(\tilde{\sigma}, \tilde{\tau}) \leq \frac{8}{m^\beta}. \quad \square$$

Observe that if $\beta > 0$, then there is $m_0 \in \mathbb{N}$ such that, for $m \geq m_0$,

$$\sum_{i=1}^{\lceil m^\beta \rceil} i \leq \frac{m^{2\beta} + 3m^\beta + 2}{2} \leq m^{2\beta}.$$

Lemma 24. *For every $\beta > 0$ there is $m_0 \in \mathbb{N}$ such that, if $m \geq m_0$, $k \in \mathbb{N}$, $n = km$ and $\tau \in S_n$ is such that $w(\tau) \leq \lceil m^\beta \rceil$, then there is $\rho \in S_n$ such that $w(\rho) \leq \lceil m^\beta \rceil$, $C_1(\rho) \supset C_1(\sigma)$, $d(\rho, \tau) < m^{2\beta-1}$ and $C_i(\rho)$ is either 0 or a multiple of k for every $i \in \mathbb{N}$.*

Proof. Define, for $i \in \{2, \dots, \lceil m^\beta \rceil\}$,

$$|C_i(\tau)| = n_i = t_i k + r_i,$$

where $0 \leq r_i < k$. Observe that

$$\sum_{i=2}^{\lceil m^\beta \rceil} i r_i \leq k \sum_{i=2}^{\lceil m^\beta \rceil} i \leq k m^{2\beta}.$$

Consider the permutation ρ obtained dropping r_i i -cycles from σ for every $i \in \{2, 3, \dots, \lceil m^\beta \rceil\}$. Observe that $|C_i(\rho)| = k t_i \in k\mathbb{N}$ for $i = 2, 3, \dots, \lceil m^\beta \rceil$. Moreover,

$$|C_1(\rho)| = km - \sum_{i=2}^{\lceil m^\beta \rceil} C_i(\rho) = \left(m - \sum_{i=2}^{\lceil m^\beta \rceil} t_i \right) k \in k\mathbb{N}$$

and $|C_i(\rho)| = 0$ for $i > \lceil m^\beta \rceil$. Finally,

$$km \cdot d(\tau, \rho) \leq \sum_{i=2}^{\lceil m^\beta \rceil} i r_i \leq k m^{2\beta}$$

and hence $d(\tau, \rho) \leq m^{2\beta-1}$. \square

Observe that if $\rho \in S_{km}$ is such that $|C_i(\sigma)| \in k\mathbb{N}$ for every $i \in \mathbb{N}$, then there is an isometric embedding $\Phi : S_m \rightarrow S_{km}$ such that ρ belongs to the image of Φ .

Proposition 25. *There is $m_0 \in \mathbb{N}$ such that, for every $m \geq m_0$ and $k \in \mathbb{N}$, if $\sigma \in S_{km}$ then there is $\rho \in S_{km}$ such that $C_1(\rho) \supset C_1(\sigma)$, $w(\rho) \leq \lceil \sqrt[3]{m} \rceil$, $d(\rho, \sigma) \leq \frac{9}{\sqrt[3]{m}}$ and $\rho = \Phi(\tilde{\rho})$ for some $\tilde{\rho} \in S_m$ and isometric embedding $\Phi : S_m \rightarrow S_{km}$*

Proof. By the first corollary, there is $\tau \in S_{km}$ such that $d(\tau, \sigma) < \frac{8}{\sqrt[3]{m}}$, $w(\tau) \leq \lceil \sqrt[3]{m} \rceil$ and $C_1(\tau) \supset C_1(\sigma)$. By the last proposition, there is $\rho \in S_{km}$ such that $C_1(\rho) \supset C_1(\tau)$, $d(\rho, \sigma) \leq \frac{1}{\sqrt[3]{m}}$, $w(\rho) \leq \lceil \sqrt[3]{m} \rceil$ and $\rho = \Phi(\tilde{\rho})$ for some $\tilde{\rho} \in S_m$ and isometric embedding $\Phi : S_m \rightarrow S_{km}$. Finally,

$$d(\sigma, \rho) \leq d(\tau, \rho) + d(\tau, \sigma) \leq \frac{1}{\sqrt[3]{m}} + \frac{8}{\sqrt[3]{m}} \leq \frac{9}{\sqrt[3]{m}}. \quad \square$$

Considering the fact that the algebraic embedding of S_{km} into S_{km+r} for $0 \leq r < m$ preserves distances up to $1 - \frac{1}{m}$, it follows from the previous proposition that the sequence $(S_n)_{n \in \mathbb{N}}$ satisfies the hypothesis of Lemma 18. This concludes the proof of the fact that all universal sofic groups have the same Σ_2 -theory.

REFERENCES

1. I. Ben Yaacov, A. Berenstein, C.W. Henson, and A. Usvyatsov, *Model theory for metric structures*, Model Theory with Applications to Algebra and Analysis, Vol. II (Z. Chatzidakis et al.eds.), London Math. Soc. Lecture Notes Series, no. 350, Cambridge University Press, 2008, pp. 315-427. MR2436146 (2009j:03061)
2. H. A. Dye, *The unitary structure in finite rings of operators*, Duke Math. J. **20** (1953), pp. 55-69. MR0052695 (14,659g)
3. G. Elek, *Connes Embeddings and von Neumann Regular Closures of Group Algebras*, preprint, arXiv:1006.5378
4. G. Elek and E. Szabó, *Hyperlinearity, essentially free actions and L_2 -invariants. The sofic property*, Math. Ann. **332** (2005), no. 2, pp. 421-441. MR2178069 (2007i:43002)

5. I. Farah, B. Hart, and D. Sherman, *Model theory of operator algebras I: Stability*, preprint, arXiv:0908.2790, 2009
6. I. Farah, B. Hart, and D. Sherman, *Model theory of operator algebras II: Model theory*, preprint, arXiv:1004.0741, 2009
7. I. Farah, S. Shelah, *A dichotomy for the number of ultrapowers*, J. Math. Log. **10** (2010), pp. 45–81
8. L. Ge and D. Hadwin, *Ultraproducts of C^* -algebras*, Recent advances in operator theory and related topics (Szeged, 1999), Oper. Theory Adv. Appl., vol. 127, Birkhäuser, Basel, 2001, pp. 305–326. MR1902808 (2003f:46083)
9. L. Glebsky, *Almost commuting matrices with respect to normalized Hilbert-Schmidt norm*, preprint, arXiv:1002.3082, 2010
10. K. R. Goodearl, *Von Neumann regular rings*, Robert E. Krieger Publishing Co., Inc., Malabar, FL, (1991). MR1150975 (93m:16006)
11. V. Pestov, *Hyperlinear and sofic groups: a brief guide*, Bull. Symbolic Logic **14** (2008), 449–480. MR2460675 (2009k:20103)
12. S. Thomas, *On the number of universal sofic groups*, Proc. Amer. Math. Soc. **138** (2010), no. 7, 2585–2590. MR2607888 (2011c:20084)
13. J. von Neumann, *Approximative properties of matrices of high finite order*, Portugal. Math. vol. 3 (1942), pp. 1–62. MR0006137

DEPARTMENT OF MATHEMATICS AND STATISTICS, N520 ROSS, 4700 KEELE STREET, TORONTO,
ONTARIO M3J 1P3

E-mail address: mlupini@mathstat.yorku.ca